

Article

Algebraic Perspective of Cubic Multi-Polar Structures on BCK/BCI-Algebras

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Abstract: Cubic multipolar structure with finite degree (briefly, cubic k -polar (C_kP) structure) is a new hybrid extension of both k -polar fuzzy (${}_kPF$) structure and cubic structure in which C_kP structure consists of two parts; the first one is an interval-valued k -polar fuzzy ($IV{}_kPF$) structure acting as a membership grade extended from the interval $\mathbb{P}[0, 1]$ to $\mathbb{P}[0, 1]^k$ (i.e., from interval-valued of real numbers to the k -tuple interval-valued of real numbers), and the second one is a ${}_kPF$ structure acting as a nonmembership grade extended from the interval $[0, 1]$ to $[0, 1]^k$ (i.e., from real numbers to the k -tuple of real numbers). This approach is based on generalized cubic algebraic structures using polarity concepts and therefore the novelty of a C_kP algebraic structure lies in its large range comparative to both ${}_kPF$ algebraic structure and cubic algebraic structure. The aim of this manuscript is to apply the theory of C_kP structure on BCK/BCI-algebras. We originate the concepts of C_kP subalgebras and (closed) C_kP ideals. Moreover, some illustrative examples and dominant properties of these concepts are studied in detail. Characterizations of a C_kP subalgebra/ideal are given, and the correspondence between C_kP subalgebras and (closed) C_kP ideals are discussed. In this regard, we provide a condition for a C_kP subalgebra to be a C_kP ideal in a BCK-algebra. In a BCI-algebra, we provide conditions for a C_kP subalgebra to be a C_kP ideal, and conditions for a C_kP subalgebra to be a closed C_kP ideal. We prove that, in weakly BCK-algebra, every C_kP ideal is a closed C_kP ideal. Finally, we establish the C_kP extension property for a C_kP ideal.

Keywords: multipolar structure; cubic multipolar structure; cubic multipolar subalgebra; cubic multipolar ideal; closed cubic multipolar ideal

MSC: 06F35; 03G25; 03B52; 03B05



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1. Introduction

In pure mathematics, an algebraic structure consists of a nonempty underlying set (carrier set or domain) \mathbb{B} , and a collection of binary operations on \mathbb{B} , satisfying some finite set of identities called axioms. Abstract algebra is primarily concerned with the study of algebraic structures and their properties. As a ramification of algebraic structures, BCK and BCI-algebras first emerged in pure mathematics in 1966, in the work by Imai and co-workers [1–3]. BCK and BCI-algebras are the algebraic formulations of the set difference and its properties in set theory, as well as the implicational functor in logical systems. BCK/BCI-algebras have been used in numerous fields; in particular, BCK-algebra is utilized in the theory of coding (see [4–6]).

Imprecision, vagueness and uncertainty are traits that are pervasive in problems circulating in the real world, and these characteristics cannot be dealt with effectively using mathematical techniques that are traditionally employed to handle vagueness and uncertainties. Some of the pioneering approaches used to handle these constraints include fuzzy sets (FSs) [7], bipolar fuzzy sets (BFSs) [8], intuitionistic fuzzy sets (IFSs) [9], interval-valued fuzzy sets (IVFSs) [10] and cubic sets (CSs) [11]. Although all of the aforementioned

approaches can handle uncertainties and ambiguities in information, none of them can handle the multi polarity that present in many real life scenarios. As a result, in [12], the k -polar fuzzy (kPF) set model was propounded, followed by the development and enhancement of the theory.

The notion of kPF sets stems from the concept of FS s, and it is an expansion of the theory of BFS s. We can note that the mathematical theories of a $2PF$ set and a BFS are equivalent. Chen et al. [12] expanded the view of BFS into a kPF set by utilizing the idea of one-to-one correspondence, and demonstrated some of its practical applications in real world situations. A kPF set assigns k independent degrees/values to each crisp object/element. A kPF structure of a nonempty universal set \mathbb{B} is a mapping $\tilde{\pi}_k : \mathbb{B} \rightarrow [0, 1]^k$, symbolized by $\tilde{\mathcal{C}}_{\tilde{\pi}_k} = \{ \langle m, \tilde{\pi}^{(j)}(m) \rangle \mid m \in \mathbb{B} \text{ and } j \in \{1, 2, \dots, k\} \}$, where, for $j \in \{1, 2, \dots, k\}$, $\tilde{\pi}^{(j)} : \mathbb{B} \rightarrow [0, 1]^k$ is the j -th projection mathematical mapping. In [9], Atanassov presented the concept of IFS using a grade of membership and a grade of non-membership which are fuzzy values to each crisp object/element. An abstraction of IFS and kPF set was inaugurated by Kang et al. [13], named as a k -polar intuitionistic fuzzy ($kPIF$) set. They used a grade of membership and a grade of non-membership which are a multi fuzzy value to each crisp object/element. As an abstraction of IFS , Jun et al. [11] displayed the concept of a CS using a grade of membership which is a fuzzy interval value and a grade of non-membership which is a fuzzy value to each crisp object/element. A CS of a nonempty universal set \mathbb{B} is a mapping $\mathcal{C}_{(\tilde{\omega}, \pi)} : \mathbb{B} \rightarrow \mathbb{P}[0, 1] \times [0, 1]$ defined by $\mathcal{C}_{(\tilde{\omega}, \pi)} = \{ \langle m, \tilde{\omega}(m), \pi(m) \rangle \mid m \in \mathbb{B} \}$, where $\tilde{\omega} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]$ and $\pi : \mathbb{B} \rightarrow [0, 1]$ are an $IVFS$ and a FS , respectively. A C_kP structure $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ of \mathbb{B} , an extension of a CS $\mathcal{C}_{(\tilde{\omega}, \pi)}$ and a kPF set $\tilde{\mathcal{C}}_{\tilde{\pi}_k}$, was propounded by Riaz and Hashmi [14] in 2019 by using a grade of membership which is a multi fuzzy interval value and a grade of non-membership which is a multi fuzzy value to each crisp object/element. They established C_kP averaging aggregation operators for agribusiness MAGDM. This concept manipulates not only multi-attributed data but also cubic data. As a result, CS and kPF set are special cases of C_kP structures. Garg et al. [15] explored some new operational laws and produced some related results of C_kP structure. They applied the idea of C_kP structure in pattern recognition and medical diagnosis. There is a relation between a C_kP structure and other hybrid FS . This relationship can be elaborated in Figure 1, where $j \in \{1, 2, \dots, k\}$.

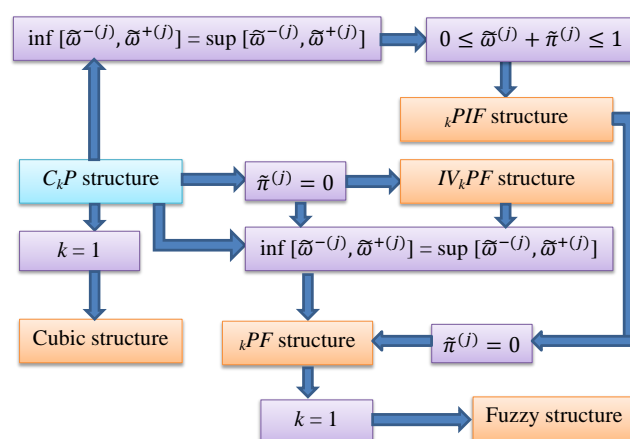


Figure 1. Relationship between a C_kP structure and other hybrid FS s.

In the fuzzification of the algebraic structures, the concept of FS was connected with BCK-algebra by Xi [16]. Merging the ideas of FS and BCI-algebra, fuzzy BCI-algebra was implemented by Ahmad [17]. Globally, interest in FS theory and its application has been growing rapidly in recent years. As a result, many authors worked on BCK/BCI algebra and ideals in FS theory (see works by the authors of [18–21]). In the polarity

of fuzziness algebraic structures, Al-Masarwah and Ahmad [22,23] applied kPF sets to BCK/BCI-algebras by merging the notions of kPF sets and BCK/BCI-algebras. They presented and investigated the concepts of kPF subalgebras, kPF (commutative) ideals and (α, β) - kPF ideals. Borzooei et al. [24] proposed and studied the polarity of generalized neutrosophic subalgebras in BCK/BCI-algebras. Kang et al. [13] applied the polarity of IFS s to BCK/BCI-algebras and investigated some related properties. In [25], Dogra and Pal defined the polarity of picture fuzzy subalgebras and (implicative) ideals in BCK-algebras and presented some related basic results. Applying the idea of interval-valued k -polar fuzzy (IV_kPF) sets to BCK/BCI-algebras, Muhiuddin and Al-Kadi extended the work of [22], proposed the concepts of IV_kPF subalgebras and IV_kPF (commutative) ideals in BCK(BCI)-algebras, and investigated some essential properties [26]. Since that time, the notions of multi-polar hybrid FS s have been utilized in many scientific and technological fields, among which are graph theory [27–30], topology [31], matroid theory [32], decision-making [33,34] and so on. By extending the work of [22] and motivated by the previous works, the notion of C_kP subalgebras and (closed) C_kP ideals in BCK/BCI-algebras is presented by combining the concept of kPF subalgebras/(closed) ideals and cubic subalgebras/(closed) ideals, and characterizations of subalgebras and ideals according to the properties of C_kP structures are given in this present work. To demonstrate the novelty of this structure, Figure 2 illustrates a novel hybrid extension of cubic BCK/BCI-algebras and kPF BCK/BCI-algebras known as C_kP BCK/BCI-algebras.

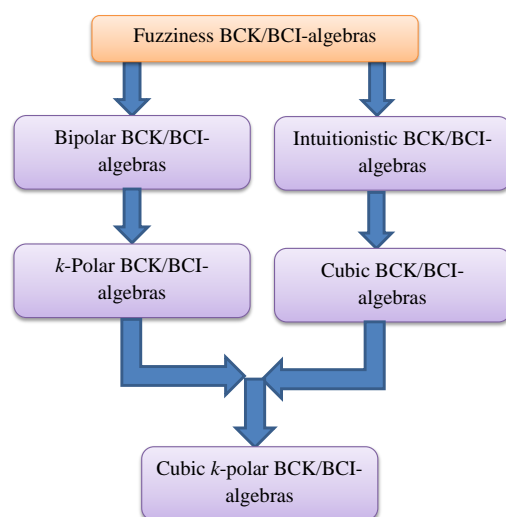


Figure 2. Contributions toward C_kP BCK/BCI-algebras.

This study is the first attempt to discuss and use the C_kP structures in algebraic structures, in particular BCK/BCI-algebras. The scheme of the whole paper is organized as follows: In Section 2, we present some basic notions and preliminary definitions with respect to BCK/BCI-algebras and C_kP structures that are used throughout the paper. In Section 3, we originate the C_kP subalgebras and (closed) C_kP ideals in BCK(BCI)-algebras. Then, we study some dominant properties of these concepts in detail. For the characterizations of BCK/BCI-algebras, we give a fundamental bridge between crisp subalgebras/ideals and kPF subalgebras/ideals. In Section 4, we discuss relations between C_kP subalgebra, C_kP ideals and closed C_kP ideals. In this regard, we provide a condition for a C_kP subalgebra to be a C_kP ideal in a BCK-algebra. In a BCI-algebra, we provide conditions for a C_kP subalgebra to be a C_kP ideal, and conditions for a C_kP subalgebra to be a closed C_kP ideal. We prove that, in weakly BCK-algebra, every C_kP ideal is a closed C_kP ideal. Some illustrative examples are given to support the results in this paper. In Section 5, we establish the C_kP extension property for a C_kP ideal. Finally, the conclusions and some future works of this paper are presented in Section 6.

2. Basic Definitions

In the current segment, we present some preliminary definitions and results related to BCK and BCI-algebras (see [1–3]) and kPF structures (see [12]) that are used throughout the present work. Henceforth, \mathbb{B} shall denote a BCK/BCI-algebra, unless otherwise specified.

An algebraic structure $(\mathbb{B}; *, 0)$ is called a BCI-algebra; if $\forall m, z, h \in \mathbb{B}$, it satisfies

$$(BI) ((m * z) * (m * h)) * (h * z) = 0,$$

$$(BII) (m * (m * z)) * z = 0,$$

$$(BIII) m * m = 0,$$

$$(BIV) m * z = 0 \text{ and } z * m = 0 \Rightarrow m = z.$$

If a BCI-algebra \mathbb{B} satisfies $0 * m = 0$ (resp., $0 * m \leq m$) for any $m \in \mathbb{B}$, then \mathbb{B} is said to be a BCK-algebra (resp., a weakly BCK-algebra (see [35])). Any BCK/BCI-algebra \mathbb{B} satisfies the following: $\forall m, z, h \in \mathbb{B}$,

$$(B1) m * 0 = m,$$

$$(B2) (m * z) * h = (m * h) * z,$$

$$(B3) m \leq z \Rightarrow m * h \leq z * h, h * z \leq h * m,$$

where $m \leq z \Leftrightarrow m * z = 0$. A subset $A (\neq \emptyset)$ of \mathbb{B} is called a subalgebra of \mathbb{B} if $m * z \in A$, $\forall m, z \in A$. A subset $K (\neq \emptyset)$ of \mathbb{B} is called an ideal of \mathbb{B} if $0 \in K$ and $\forall m, z \in K$, $m * z \in K$ and $z \in K \Rightarrow m \in K$. An ideal $K (\neq \emptyset)$ of a BCI-algebra \mathbb{B} is called a closed ideal of \mathbb{B} (see [35]) if $\forall m \in K \Rightarrow 0 * m \in K$ for any $m \in \mathbb{B}$.

A BCI-algebra \mathbb{B} is said to be p-semisimple (see [35]) if $(0 * (0 * m)) = m, \forall m \in \mathbb{B}$. In a p-semisimple BCI-algebra \mathbb{B} , the following holds: $\forall m, z \in \mathbb{B}$,

$$(P1) (0 * (m * z)) = z * m,$$

$$(P2) m * (m * z) = z.$$

A BCI-algebra \mathbb{B} is said to be an associative (see [35]) if $(m * z) * h = m * (z * h)$, $\forall m, z, h \in \mathbb{B}$.

By the interval number \tilde{z} , we mean a subinterval defined as $[z^-, z^+]$ of $[0, 1]$, where $0 \leq z^- \leq z^+ \leq 1$. The set of all interval numbers is denoted by $\mathbb{P}[0, 1]$. The interval $[z, z]$ indicated by the number $z \in [0, 1]$ for whatever follows. For the interval numbers $\tilde{z}_i = [z_i^-, z_i^+], \tilde{h}_i = [h_i^-, h_i^+] \in \mathbb{P}[0, 1]$, where $i \in \Lambda$. We describe

- $\tilde{z}_i \wedge \tilde{h}_i = [z_i^- \wedge h_i^-, z_i^+ \wedge h_i^+]$,
- $\tilde{z}_i \vee \tilde{h}_i = [z_i^- \vee h_i^-, z_i^+ \vee h_i^+]$,
- $\tilde{z}_1 \preceq \tilde{z}_2 \Leftrightarrow z_1^- \leq z_2^- \text{ and } z_1^+ \leq z_2^+$,
- $\tilde{z}_1 = \tilde{z}_2 \Leftrightarrow z_1^- = z_2^- \text{ and } z_1^+ = z_2^+$.

To say that $\tilde{z}_1 \prec \tilde{z}_2$ (resp. $\tilde{z}_1 \succ \tilde{z}_2$), we mean $\tilde{z}_1 \preceq \tilde{z}_2$ and $\tilde{z}_1 \neq \tilde{z}_2$ (resp. $\tilde{z}_1 \succeq \tilde{z}_2$ and $\tilde{z}_1 \neq \tilde{z}_2$).

Definition 1 ([10]). Let \mathbb{B} be a nonempty set. An IVFS of \mathbb{B} is a mapping

$$\tilde{\omega} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]$$

defined as

$$\tilde{\omega} = \{ \langle m, \omega^-(m), \omega^+(m) \rangle \mid m \in \mathbb{B} \},$$

where $\omega^- : \mathbb{B} \rightarrow [0, 1]$ and $\omega^+ : \mathbb{B} \rightarrow [0, 1]$ are FSs on \mathbb{B} .

Definition 2 ([14]). Let \mathbb{B} be a non-empty set. A C_kP structure of \mathbb{B} is a mapping

$$\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]^k \times [0, 1]^k$$

defined as

$$\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k} = \left\{ \left\langle m, \left(\tilde{\omega}^{(j)}(m), \tilde{\pi}^{(j)}(m) \right) \right\rangle \mid m \in \mathbb{B} \text{ and } j \in \{1, 2, \dots, k\} \right\},$$

where for $j \in \{1, 2, \dots, k\}$, $\tilde{\omega}^{(j)} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]^k$ and $\tilde{\pi}^{(j)} : \mathbb{B} \rightarrow [0, 1]^k$ are the j -th projection mappings.

That is,

$$\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k} = \left\{ \left\langle m, \left([\tilde{\omega}^{-(j)}(m), \tilde{\omega}^{+(j)}(m)], \tilde{\pi}^{(j)}(m) \right) \right\rangle \mid m \in \mathbb{B} \text{ and } j \in \{1, 2, \dots, k\} \right\},$$

where $\tilde{\omega}^{-(j)}$, $\tilde{\omega}^{+(j)}$ and $\tilde{\pi}^{(j)}$ are FSs of \mathbb{B} with $\tilde{\omega}^{-(j)} \leq \tilde{\omega}^{+(j)}$ for all $j \in \{1, 2, \dots, k\}$. The complement of a C_kP structure $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is $(\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k})^c$, where

$$(\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k})^c = \left\{ \left\langle m, \left(1 - \tilde{\omega}^{(j)}(m), 1 - \tilde{\pi}^{(j)}(m) \right) \right\rangle \mid m \in \mathbb{B} \text{ and } j \in \{1, 2, \dots, k\} \right\}.$$

That is,

$$(\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k})^c = \left\{ \left\langle m, \left([1 - \tilde{\omega}^{-(j)}(m), 1 - \tilde{\omega}^{+(j)}(m)], 1 - \tilde{\pi}^{(j)}(m) \right) \right\rangle \mid m \in \mathbb{B} \text{ and } j \in \{1, 2, \dots, k\} \right\}.$$

3. Cubic k -Polar Subalgebras and Ideals

In this section, we apply the finite polarity of CSs to subalgebras and ideals of BCK/BCI-algebras. We introduce the concepts of a C_kP subalgebra and a C_kP ideal, and discuss their characterizations.

Definition 3. A C_kP structure $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ of \mathbb{B} is called a C_kP subalgebra if for all $m, z \in \mathbb{B}$,

- (1) $\tilde{\omega}(m * z) \succeq \tilde{\omega}(m) \bar{\wedge} \tilde{\omega}(z)$,
- (2) $\tilde{\pi}(m * z) \leq \tilde{\pi}(m) \vee \tilde{\pi}(z)$.

That is,

- (1) $\tilde{\omega}^{(j)}(m * z) \succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z)$,
- (2) $\tilde{\pi}^{(j)}(m * z) \leq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)$

for all $j \in \{1, 2, \dots, k\}$.

Example 1. Let $\mathbb{B} = \{0, u, v, \varepsilon\}$ be a set with the $*$ -operation given by Table 1:

Table 1. Cayley table of operation $*$.

$*$	0	u	v	ε
0	0	0	0	0
u	u	0	0	u
v	v	u	0	v
ε	ε	ε	ε	0

Then, $(\mathbb{B}; *, 0)$ is a BCK-algebra. Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_4} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]^4 \times [0, 1]^4$ be a C_4P structure on \mathbb{B} defined by:

$$\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_4} = \left\{ \begin{array}{l} \left\langle 0, \left([0.33, 0.83], 0.27 \right), \left([0.30, 0.80], 0.21 \right), \left([0.70, 0.80], 0.12 \right), \left([0.20, 0.25], 0.10 \right) \right\rangle, \\ \left\langle u, \left([0.15, 0.56], 0.35 \right), \left([0.10, 0.50], 0.42 \right), \left([0.30, 0.50], 0.41 \right), \left([0.10, 0.15], 0.15 \right) \right\rangle, \\ \left\langle v, \left([0.33, 0.83], 0.27 \right), \left([0.30, 0.80], 0.39 \right), \left([0.10, 0.80], 0.44 \right), \left([0.05, 0.10], 0.20 \right) \right\rangle, \\ \left\langle \varepsilon, \left([0.15, 0.56], 0.35 \right), \left([0.10, 0.50], 0.33 \right), \left([0.50, 0.50], 0.32 \right), \left([0.10, 0.15], 0.15 \right) \right\rangle. \end{array} \right\}$$

Then, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_4}$ is a C_4P subalgebra of \mathbb{B} .

Lemma 1. If $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra over \mathbb{B} , then

$$\tilde{\omega}^{(j)}(0) \succeq \tilde{\omega}^{(j)}(m) \text{ and } \tilde{\pi}^{(j)}(0) \leq \tilde{\pi}^{(j)}(m) \quad (1)$$

for all $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$.

Proof. For every $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, m\}$, we have

$$\tilde{\omega}^{(j)}(0) = \tilde{\omega}^{(j)}(m * m) \succeq \tilde{\omega}^{(j)}(m) \wedge \tilde{\omega}^{(j)}(m) = \tilde{\omega}^{(j)}(m)$$

and

$$\tilde{\pi}^{(j)}(0) = \tilde{\pi}^{(j)}(m * m) \leq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(m) = \tilde{\pi}^{(j)}(m).$$

□

Theorem 1. Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP subalgebra over \mathbb{B} . For a given $m \in \mathbb{B}$ and $\forall j \in \{1, 2, \dots, m\}$, the following are equivalent:

- (1) $\tilde{\omega}^{(j)}(m) = \tilde{\omega}^{(j)}(0)$ and $\tilde{\pi}^{(j)}(m) = \tilde{\pi}^{(j)}(0)$.
- (2) $\tilde{\omega}^{(j)}(z) \preceq \tilde{\omega}^{(j)}(m * z)$ and $\tilde{\pi}^{(j)}(z) \geq \tilde{\pi}^{(j)}(m * z)$ for any $z \in \mathbb{B}$ with $m * z \in \mathbb{B}$.

Proof. Assume that (1) is true. Then, for any $j \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned} \tilde{\omega}^{(j)}(z) &= \tilde{\omega}^{(j)}(0) \wedge \tilde{\omega}^{(j)}(z) \\ &= \tilde{\omega}^{(j)}(m) \wedge \tilde{\omega}^{(j)}(z) \\ &\preceq \tilde{\omega}^{(j)}(m * z) \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}^{(j)}(z) &= \tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(z) \\ &= \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z) \\ &\geq \tilde{\pi}^{(j)}(m * z). \end{aligned}$$

Conversely, assume that (2) is valid, since $m * 0 = m$ for all $m \in \mathbb{B}$. Taking $z = 0$, we have $\tilde{\omega}^{(j)}(0) \preceq \tilde{\omega}^{(j)}(m * 0) = \tilde{\omega}^{(j)}(m)$ and $\tilde{\pi}^{(j)}(0) \geq \tilde{\pi}^{(j)}(m * 0) = \tilde{\pi}^{(j)}(m)$ for any $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, m\}$. It follows from Lemma 1 that we have (1). □

Theorem 2. A C_kP structure $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ over \mathbb{B} is a C_kP subalgebra if and only if $\tilde{\omega}$ is an IV_kPF subalgebra and $\tilde{\pi}^c$ is a kPF subalgebra of \mathbb{B} , where $\tilde{\pi}^c = 1 - \tilde{\pi}$ i.e., $\tilde{\pi}^{c(j)} = 1 - \tilde{\pi}^{(j)}$.

Proof. Suppose that $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra over \mathbb{B} . Then, it is clear that $\tilde{\omega}$ is an IV_kPF subalgebra of \mathbb{B} . For every $m, z \in \mathbb{B}$, we have

$$\begin{aligned} \tilde{\pi}^{c(j)}(m * z) &= 1 - \tilde{\pi}^{(j)}(m * z) \\ &\geq 1 - (\tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)) \\ &= (1 - \tilde{\pi}^{(j)}(m)) \wedge (1 - \tilde{\pi}^{(j)}(z)) \\ &= \tilde{\pi}^{c(j)}(m) \wedge \tilde{\pi}^{c(j)}(z) \end{aligned}$$

for each $j \in \{1, 2, \dots, k\}$. Thus, $\tilde{\pi}^c$ is a k PF subalgebra of \mathbb{B} .

Conversely, assume that $\tilde{\omega}$ is an IV_k PF subalgebra and $\tilde{\pi}^c$ is a k PF subalgebra of \mathbb{B} , and let $m, z \in \mathbb{B}$. Having $\tilde{\omega}$ as an IV_k PF subalgebra of \mathbb{B} , implies $\tilde{\omega}^{(j)}(m * z) \succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z)$, for each $j \in \{1, 2, \dots, k\}$. Moreover, we obtain

$$\begin{aligned} 1 - \tilde{\pi}^{(j)}(m * z) &= \tilde{\pi}^{c(j)}(m * z) \\ &\geq \tilde{\pi}^{c(j)}(m) \wedge \tilde{\pi}^{c(j)}(z) \\ &= (1 - \tilde{\pi}^{(j)}(m)) \wedge (1 - \tilde{\pi}^{(j)}(z)) \\ &= 1 - (\tilde{\pi}^{c(j)}(m) \vee \tilde{\pi}^{c(j)}(z)). \end{aligned}$$

That is, $\tilde{\pi}^{(j)}(m * z) \leq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)$ for each $j \in \{1, 2, \dots, k\}$. Therefore, $\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_k P subalgebra of \mathbb{B} . \square

Definition 4. Let $\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}$ be any C_k P structure over \mathbb{B} . Then, $([\widetilde{u, v}], \tilde{\gamma})$ -level of $\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}$ is the crisp set in \mathbb{B} denoted by $\Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma}))$ and is defined as

$$\Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma})) = \{m \in \mathbb{B} \mid \tilde{\omega}(m) \succeq [\widetilde{u, v}], \tilde{\pi}(m) \leq \tilde{\gamma}\},$$

where $[\widetilde{u, v}] \in \mathbb{P}[0, 1]^k$ and $\tilde{\gamma} \in [0, 1]^k$.

That is,

$$\Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma})) = \{m \in \mathbb{B} \mid \tilde{\omega}^{(j)}(m) \succeq [u_j, v_j], \tilde{\pi}^{(j)}(m) \leq \gamma_j, \forall j \in \{1, 2, \dots, k\}\},$$

where $[u_j, v_j] \in \mathbb{P}[0, 1]$ and $\gamma_j \in [0, 1]$ for each $j \in \{1, 2, \dots, k\}$.

Theorem 3. Let $\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_k P subalgebras over \mathbb{B} . Then, $\Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma}))$ is a crisp subalgebra of \mathbb{B} .

Proof. Let $m, z \in \Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma}))$. Then, $\tilde{\omega}^{(j)}(m) \succeq [u_j, v_j]$, $\tilde{\pi}^{(j)}(m) \leq \gamma_j$, $\tilde{\omega}^{(j)}(z) \succeq [u_j, v_j]$ and $\tilde{\pi}^{(j)}(z) \leq \gamma_j$ for each $j \in \{1, 2, \dots, k\}$. Having $\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}$ as a C_k P subalgebras of \mathbb{B} implies

$$\begin{aligned} \tilde{\omega}^{(j)}(m * z) &\succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &\succeq [u_j, v_j] \bar{\wedge} [u_j, v_j] \\ &= [u_j, v_j] \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}^{(j)}(m * z) &\leq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z) \\ &\leq \gamma_j \vee \gamma_j \\ &= \gamma_j. \end{aligned}$$

Thus, $m, z \in \Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma})) \Rightarrow m * z \in \Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma}))$. Thus, $\Theta(\mathcal{E}_{(\tilde{\omega}, \tilde{\pi})_k}; ([\widetilde{u, v}], \tilde{\gamma}))$ is a crisp subalgebra of \mathbb{B} . \square

Theorem 4. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP structure over \mathbb{B} . Then, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of \mathbb{B} if all $([u, v], \tilde{\gamma})$ -levels of $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ are crisp subalgebras of \mathbb{B} .

Proof. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP structure over \mathbb{B} such that all $([u, v], \tilde{\gamma})$ -levels of $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ are crisp subalgebras of \mathbb{B} for each $[u, v] \in \mathbb{P}[0, 1]^k$ and $\tilde{\gamma} \in [0, 1]^k$. On the contrary, let $\tilde{\omega}^{(j)}(m * z) \prec \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z)$ and $\tilde{\pi}^{(j)}(m * z) > \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)$ for some $m, z \in \mathbb{B}$. Thus, there exist $[y, q] \in \mathbb{P}[0, 1]^k$ and $\tilde{\delta} \in [0, 1]^k$ such that $\tilde{\omega}^{(j)}(m * z) \prec [y, q] \preceq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z)$ or $\tilde{\pi}^{(j)}(m * z) > \tilde{\delta} \geq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)$ for each $j \in \{1, 2, \dots, k\}$, which implies $m, z \in \Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma}))$, but $m * z \notin \Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma}))$. This is a contradiction and hence $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of \mathbb{B} . \square

Definition 5. A C_kP structure $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ over \mathbb{B} is called a C_kP ideal of \mathbb{B} if it satisfies the condition (1) of Lemma 1 and for all $m, z \in \mathbb{B}$,

- (1) $\tilde{\omega}(m) \succeq \tilde{\omega}(m * z) \bar{\wedge} \tilde{\omega}(z)$,
- (2) $\tilde{\pi}(m) \leq \tilde{\pi}(m * z) \vee \tilde{\pi}(z)$.

That is,

- (1) $\tilde{\omega}^{(j)}(m * z) \succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z)$,
- (2) $\tilde{\pi}^{(j)}(m * z) \leq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)$

for all $j \in \{1, 2, \dots, k\}$.

Now, we are going to discuss some significant results on a C_kP ideal of \mathbb{B} .

Theorem 5. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of \mathbb{B} . Then,

$$\tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(z) \text{ and } \tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(z)$$

for all $m, z \in \mathbb{B}$ with $m \leq z$ and $\forall j \in \{1, 2, \dots, k\}$.

Proof. Let $m, z \in \mathbb{B}$ with $m \leq z$. Then, $m * z = 0$. Having $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ as a C_kP ideal of \mathbb{B} implies

$$\begin{aligned} \tilde{\omega}^{(j)}(m) &\succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &= \tilde{\omega}^{(j)}(0) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &= \tilde{\omega}^{(j)}(z) \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}^{(j)}(m * z) &\leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) \\ &= \tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(z) \\ &= \tilde{\pi}^{(j)}(z) \end{aligned}$$

for all $j \in \{1, 2, \dots, k\}$. This completes the proof. \square

Theorem 6. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of \mathbb{B} . Then,

$$\tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(z) \bar{\wedge} \tilde{\omega}^{(j)}(w) \text{ and } \tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(z) \vee \tilde{\pi}^{(j)}(w)$$

for all $m, z, w \in \mathbb{B}$ with $m * z \leq w$ and $\forall j \in \{1, 2, \dots, k\}$.

Proof. Let $m, z, w \in \mathbb{B}$ with $m * z \leq w$. Then, $(m * z) * w = 0$. Having $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ as a C_kP ideal of \mathbb{B} implies

$$\begin{aligned}\tilde{\omega}^{(j)}(m) &\succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &\succeq \tilde{\omega}^{(j)}((m * z) * w) \bar{\wedge} \tilde{\omega}^{(j)}(w) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &= \tilde{\omega}^{(j)}(0) \bar{\wedge} \tilde{\omega}^{(j)}(w) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &= \tilde{\omega}^{(j)}(z) \bar{\wedge} \tilde{\omega}^{(j)}(w)\end{aligned}$$

and

$$\begin{aligned}\tilde{\pi}^{(j)}(m) &\leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) \\ &\leq \tilde{\pi}^{(j)}((m * z) * w) \vee \tilde{\pi}^{(j)}(w) \vee \tilde{\pi}^{(j)}(z) \\ &= \tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(w) \vee \tilde{\pi}^{(j)}(z) \\ &= \tilde{\pi}^{(j)}(z) \vee \tilde{\pi}^{(j)}(w)\end{aligned}$$

for all $j \in \{1, 2, \dots, k\}$. This completes the proof. \square

The subsequent theorem is an extension of Theorem 6.

Theorem 7. If $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} , then

$$\prod_{i=1}^n m * z_i = 0 \Rightarrow \begin{pmatrix} \tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(z_1) \bar{\wedge} \tilde{\omega}^{(j)}(z_2) \dots \bar{\wedge} \tilde{\omega}^{(j)}(z_n), \\ \tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(z_1) \vee \tilde{\pi}^{(j)}(z_2) \dots \vee \tilde{\pi}^{(j)}(z_n) \end{pmatrix} \quad (2)$$

for all $m, z_1, z_2, \dots, z_n \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, where $\prod_{i=1}^n m * z_i = (\dots((m * z_1) * z_2) * \dots) * z_n$.

Proof. The proof by induction on n . Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of \mathbb{B} . Theorem 6 shows that condition (2) is true for $n = 2$. Suppose that $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ satisfies the condition (2) for $n = k$, that is, for all $m, z_1, z_2, \dots, z_n \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, we have

$$\prod_{i=1}^k m * z_i = 0 \Rightarrow \begin{pmatrix} \tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(z_1) \bar{\wedge} \tilde{\omega}^{(j)}(z_2) \dots \bar{\wedge} \tilde{\omega}^{(j)}(z_k), \\ \tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(z_1) \vee \tilde{\pi}^{(j)}(z_2) \dots \vee \tilde{\pi}^{(j)}(z_k). \end{pmatrix}$$

Now, let $m, z_1, z_2, \dots, z_k, z_{k+1} \in \mathbb{B}$ such that $\prod_{i=1}^{k+1} m * z_i = 0$. Then,

$$\tilde{\omega}^{(j)}(m * z_1) \succeq \tilde{\omega}^{(j)}(z_2) \bar{\wedge} \tilde{\omega}^{(j)}(z_3) \dots \bar{\wedge} \tilde{\omega}^{(j)}(z_{k+1})$$

and

$$\tilde{\pi}^{(j)}(m * z_1) \leq \tilde{\pi}^{(j)}(z_2) \vee \tilde{\pi}^{(j)}(z_3) \dots \vee \tilde{\pi}^{(j)}(z_{k+1}).$$

for any $j \in \{1, 2, \dots, k\}$. Since $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} . It follows from Definition 5 that

$$\begin{aligned}\tilde{\omega}^{(j)}(m) &\succeq \tilde{\omega}^{(j)}(m * z_1) \bar{\wedge} \tilde{\omega}^{(j)}(z_1) \\ &\succeq \tilde{\omega}^{(j)}(z_1) \bar{\wedge} \tilde{\omega}^{(j)}(z_2) \bar{\wedge} \tilde{\omega}^{(j)}(z_3) \dots \bar{\wedge} \tilde{\omega}^{(j)}(z_{k+1})\end{aligned}$$

and

$$\begin{aligned}\tilde{\pi}^{(j)}(m) &\leq \tilde{\pi}^{(j)}(m * z_1) \vee \tilde{\pi}^{(j)}(z_1) \\ &\leq \tilde{\pi}^{(j)}(z_1) \vee \tilde{\pi}^{(j)}(z_2) \vee \tilde{\pi}^{(j)}(z_3) \dots \vee \tilde{\pi}^{(j)}(z_{k+1})\end{aligned}$$

for any $j \in \{1, 2, \dots, k\}$. This completes the proof. \square

Next, we provide characterizations of a C_kP ideal.

Theorem 8. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of \mathbb{B} . Then, $\Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$ is a crisp ideal of \mathbb{B} , provided that $\tilde{\omega}^{(j)}(0) \succeq [u_j, v_j]$ and $\tilde{\pi}^{(j)}(0) \leq \gamma_j$ for all $j \in \{1, 2, \dots, k\}$.

Proof. Clearly, $\Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$ contains at least one element. Let $m * z, z \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$. Then, $\tilde{\omega}^{(j)}(m * z) \succeq [u_j, v_j]$, $\tilde{\pi}^{(j)}(m * z) \leq \gamma_j$, $\tilde{\omega}^{(j)}(z) \succeq [u_j, v_j]$ and $\tilde{\pi}^{(j)}(m) \leq \gamma_j$ for each $j \in \{1, 2, \dots, k\}$. Having $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ as a C_kP ideal of \mathbb{B} implies

$$\begin{aligned}\tilde{\omega}^{(j)}(m) &\succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) \\ &\succeq [u_j, v_j] \bar{\wedge} [u_j, v_j] \\ &= [u_j, v_j]\end{aligned}$$

and

$$\begin{aligned}\tilde{\pi}^{(j)}(m) &\leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) \\ &\leq \gamma_j \vee \gamma_j \\ &= \gamma_j.\end{aligned}$$

for all $j \in \{1, 2, \dots, k\}$. Thus, $m * z, z \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right) \Rightarrow m \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$. Thus, $\Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$ is a crisp ideal of \mathbb{B} . \square

Theorem 9. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP structure of \mathbb{B} . Then, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} if all $([u, v], \tilde{\gamma})$ -levels of $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ are crisp ideals of \mathbb{B} .

Proof. Let $m \in \mathbb{B}$ and let $\tilde{\omega}^{(j)}(m) = [u_j, v_j]$ and $\tilde{\pi}^{(j)}(m) = \gamma_j$ for all $j \in \{1, 2, \dots, k\}$. Clearly, $[u_j, v_j] \in \mathbb{P}[0, 1]$ and $\gamma_j \in [0, 1]$ for all $j \in \{1, 2, \dots, k\}$. Now, $\tilde{\omega}^{(j)}(m) = [u_j, v_j] \preceq \tilde{\omega}^{(j)}(0)$ and $\tilde{\pi}^{(j)}(m) = \gamma_j \geq \tilde{\pi}^{(j)}(0)$ for all $j \in \{1, 2, \dots, k\}$. Since m is an arbitrary element of \mathbb{B} , therefore $\tilde{\omega}^{(j)}(m) \preceq \tilde{\omega}^{(j)}(0)$ and $\tilde{\pi}^{(j)}(m) \geq \tilde{\pi}^{(j)}(0)$ for all $j \in \{1, 2, \dots, k\}$ for all $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$. Again, let $m, z \in \mathbb{B}$, and let $\tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) = [u_j, v_j]$ and $\tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) = \gamma_j$ for all $j \in \{1, 2, \dots, k\}$. Now, $\tilde{\omega}^{(j)}(m * z) \succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) = [u_j, v_j]$ and $\tilde{\pi}^{(j)}(m * z) \leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) = \gamma_j$ for all $j \in \{1, 2, \dots, k\}$. In addition, $\tilde{\omega}^{(j)}(z) \succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) = [u_j, v_j]$ and $\tilde{\pi}^{(j)}(z) \leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) = \gamma_j$ for all $j \in \{1, 2, \dots, k\}$. Thus, $m * z, z \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$. Since $\Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$ is a crisp ideal of \mathbb{B} , therefore $m * z \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$ and $z \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right) \Rightarrow m \in \Theta\left(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})\right)$. Therefore, $\tilde{\omega}^{(j)}(m) \succeq [u_j, v_j] = \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z)$ and $\tilde{\pi}^{(j)}(m) \leq \gamma_j = \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z)$ for all $j \in \{1, 2, \dots, k\}$. Since m, z are arbitrary

elements of \mathbb{B} , therefore $\tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z)$ and $\tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z)$ for all $m, z \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$. Hence, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} . \square

Theorem 10. A C_kP structure $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ of \mathbb{B} is a C_kP ideal if and only if $\tilde{\omega}$ is an IV_kPF ideal and $\tilde{\pi}^c$ is a kPF ideal of \mathbb{B} , where $\tilde{\pi}^c = 1 - \tilde{\pi}$ i.e., $\tilde{\pi}^c(j) = 1 - \tilde{\pi}^{(j)} \forall j \in \{1, 2, \dots, k\}$.

Proof. Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of \mathbb{B} . Then, it is clear that $\tilde{\omega}$ is an IV_kPF ideal of \mathbb{B} . For every $m, z \in \mathbb{B}$, we have

$$\tilde{\pi}^{c(j)}(0) = 1 - \tilde{\pi}^{(j)}(0) \geq 1 - \tilde{\pi}^{(j)}(m) = \tilde{\pi}^{c(j)}(m)$$

and

$$\begin{aligned} \tilde{\pi}^{c(j)}(m) &= 1 - \tilde{\pi}^{(j)}(m) \\ &\geq 1 - (\tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z)) \\ &= (1 - \tilde{\pi}^{(j)}(m * z)) \wedge (1 - \tilde{\pi}^{(j)}(z)) \\ &= \tilde{\pi}^{c(j)}(m * z) \wedge \tilde{\pi}^{c(j)}(z) \end{aligned}$$

for each $j \in \{1, 2, \dots, k\}$. Thus, $\tilde{\pi}^c$ is a kPF ideal of \mathbb{B} .

Conversely, assume that $\tilde{\omega}$ is an IV_kPF ideal and $\tilde{\pi}^c$ is a kPF ideal of \mathbb{B} . For every $m, z \in \mathbb{B}$, we have $\tilde{\omega}^{(j)}(0) \succeq \tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z)$ for each $j \in \{1, 2, \dots, k\}$. Moreover, we obtain $1 - \tilde{\pi}^{(j)}(0) = \tilde{\pi}^{c(j)}(0) \geq \tilde{\pi}^{c(j)}(m) = 1 - \tilde{\pi}^{(j)}(m)$, i.e., $\tilde{\pi}^{(j)}(0) \leq \tilde{\pi}^{(j)}(m)$ and

$$\begin{aligned} 1 - \tilde{\pi}^{(j)}(m) &= \tilde{\pi}^{c(j)}(m) \\ &\geq \tilde{\pi}^{c(j)}(m * z) \wedge \tilde{\pi}^{c(j)}(z) \\ &= (1 - \tilde{\pi}^{(j)}(m * z)) \wedge (1 - \tilde{\pi}^{(j)}(z)) \\ &= 1 - (\tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z)). \end{aligned}$$

That is, $\tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z)$ for each $j \in \{1, 2, \dots, k\}$. Therefore, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} . \square

4. Correspondence between Cubic k -Polar Subalgebras and (Closed) Cubic k -Polar Ideals

In the current section, we discuss the correspondence between C_kP subalgebras and (closed) C_kP ideals.

Theorem 11. In a BCK-algebra, every C_kP ideal is a C_kP subalgebra.

Proof. Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of a BCK-algebra \mathbb{B} . Then,

$$\begin{aligned} \tilde{\omega}^{(j)}(m * z) &\succeq \tilde{\omega}^{(j)}((m * z) * m) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &= \tilde{\omega}^{(j)}((m * m) * z) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &= \tilde{\omega}^{(j)}(0 * z) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &= \tilde{\omega}^{(j)}(0) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &\succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z) \end{aligned}$$

and

$$\begin{aligned}
 \tilde{\pi}^{(j)}(m * z) &\leq \tilde{\pi}^{(j)}((m * z) * m) \vee \tilde{\pi}^{(j)}(m) \\
 &= \tilde{\pi}^{(j)}((m * m) * z) \vee \tilde{\pi}^{(j)}(m) \\
 &= \tilde{\pi}^{(j)}(0 * m) \vee \tilde{\pi}^{(j)}(m) \\
 &= \tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(m) \\
 &\leq \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z)
 \end{aligned}$$

for all $m, z \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$. Hence, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of \mathbb{B} . \square

Corollary 1. Theorem 11 is also true in a weakly BCK-algebra.

In the next example, we can note that the reverse direction of the above theorem does not have a role in general.

Example 2. Let us assume the BCK-algebra given in Example 1 and a C_3P subalgebra $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_3} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]^3 \times [0, 1]^3$ as follows:

$$\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_3} = \left\{ \begin{array}{l} \langle 0, \quad ([0.20, 0.25], 0.30), ([0.30, 0.35], 0.20), ([0.40, 0.45], 0.10) \rangle, \\ \langle u, \quad ([0.10, 0.15], 0.40), ([0.20, 0.25], 0.30), ([0.30, 0.30], 0.20) \rangle, \\ \langle v, \quad ([0.20, 0.25], 0.30), ([0.30, 0.35], 0.20), ([0.40, 0.45], 0.10) \rangle, \\ \langle \varepsilon, \quad ([0.10, 0.15], 0.40), ([0.20, 0.25], 0.30), ([0.30, 0.30], 0.20) \rangle. \end{array} \right\}$$

Here,

$$[0.10, 0.15] = \tilde{\omega}^{(1)}(u) \not\leq \tilde{\omega}^{(1)}(u * v) \wedge \tilde{\omega}^{(1)}(v) = [0.20, 0.25],$$

$$[0.20, 0.25] = \tilde{\omega}^{(2)}(u) \not\leq \tilde{\omega}^{(2)}(u * v) \wedge \tilde{\omega}^{(2)}(v) = [0.30, 0.35]$$

$$[0.30, 0.30] = \tilde{\omega}^{(3)}(u) \not\leq \tilde{\omega}^{(3)}(u * v) \wedge \tilde{\omega}^{(3)}(v) = [0.40, 0.45].$$

Thus, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_3}$ is not a C_3P ideal of \mathbb{B} although it is a C_3P subalgebra.

Theorem 12 states under which condition a C_kP subalgebra is a C_kP ideal.

Theorem 12. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP subalgebra of a BCK-algebra \mathbb{B} . Then, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal if, for all $m, z, w \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, $m * z \leq w \Rightarrow \tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(z) \wedge \tilde{\omega}^{(j)}(w)$ and $\tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(z) \vee \tilde{\pi}^{(j)}(w)$.

Proof. By given conditions, for all $m, z, w \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, $m * z \leq w \Rightarrow \tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(z) \wedge \tilde{\omega}^{(j)}(w)$ and $\tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(z) \vee \tilde{\pi}^{(j)}(w)$. Since \mathbb{B} is a BCK-algebra, then $m * (m * z) \leq z$. Thus, we obtain $\tilde{\omega}^{(j)}(m) \succeq \tilde{\omega}^{(j)}(m * z) \wedge \tilde{\omega}^{(j)}(z)$ and $\tilde{\pi}^{(j)}(m) \leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z)$ for all $j \in \{1, 2, \dots, k\}$. Thus, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} . \square

In general, the reverse direction of Theorem 11 is not established even in a BCI-algebra. We want to give a condition for the reverse direction of Theorem 11 to be established in a BCI-algebra.

Theorem 13. Every C_kP subalgebra of a p -semisimple BCI-algebra \mathbb{B} is a C_kP ideal.

Proof. Let \mathbb{B} be p-semisimple BCI-algebra and let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP subalgebra of \mathbb{B} . Let $m, z \in \mathbb{B}$. Then, $(z * m) * (0 * m) \leq z$. Since \mathbb{B} is a p-semisimple BCI-algebra, then any element in \mathbb{B} is minimal. Thus, $(z * m) * (0 * m) = z$. Since $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of \mathbb{B} and from condition (1) of Lemma 1, we have

$$\begin{aligned}\tilde{\omega}^{(j)}(z) &= \tilde{\omega}^{(j)}((z * m) * (0 * m)) \\ &\succeq \tilde{\omega}^{(j)}(z * m) \bar{\wedge} \tilde{\omega}^{(j)}(0 * m) \\ &\succeq \tilde{\omega}^{(j)}(z * m) \bar{\wedge} (\tilde{\omega}^{(j)}(0) \bar{\wedge} \tilde{\omega}^{(j)}(m)) \\ &= \tilde{\omega}^{(j)}(z * m) \bar{\wedge} \tilde{\omega}^{(j)}(m)\end{aligned}$$

and

$$\begin{aligned}\tilde{\pi}^{(j)}(m * z) &= \tilde{\pi}^{(j)}((z * m) * (0 * m)) \\ &\leq \tilde{\pi}^{(j)}(z * m) \vee \tilde{\pi}^{(j)}(0 * m) \\ &\leq \tilde{\pi}^{(j)}(z * m) \vee (\tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(m)) \\ &= \tilde{\pi}^{(j)}(z * m) \vee \tilde{\pi}^{(j)}(m)\end{aligned}$$

for all $j \in \{1, 2, \dots, k\}$. Hence, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} . \square

Theorem 11 is not generally correct in a BCI-algebra. For this, let us examine the example below.

Example 3. Let $(\mathbb{Z}, +)$ be the additive group of integers and $(\mathbb{Z}, +, 0)$ be the disjoint BCI-algebra of \mathbb{Z} . If we take $(\mathbb{B}, *, 0)$ a BCI-algebra, then the direct product of \mathbb{B} and \mathbb{Z} , denoted by $\mathbb{D} = \mathbb{B} \times \mathbb{Z}$, is a BCI-algebra. Put $J = \mathbb{B} \times (\mathbb{N} \cup \{0\})$, where \mathbb{N} is the set of all natural numbers, then J is an ideal of \mathbb{D} . Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]^k \times [0, 1]^k$ be a C_kP structure over \mathbb{D} given by:

$$\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}(m) = \left\{ \begin{array}{ll} (\widetilde{[r, 0.70]}, \tilde{l}), & m \in J, \\ (\widetilde{[0, 0.20]}, \tilde{1}), & m \notin J, \end{array} \right\}$$

where $\widetilde{[r, 0.70]} = ([r_1, 0.70], [r_2, 0.70], \dots, [r_k, 0.70]) \in \mathbb{P}(0, 1)^k$ and $\tilde{l} = (l_1, l_2, \dots, l_k) \in (0, 1)^k$. Then, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{D} . If we take $m = (0, 0) \in J$ and $z = (0, 1) \in J$, then $m * z = (0, -1) \notin J$. Hence,

$$\tilde{0} = \tilde{\omega}^{-(j)}(m * z) < \tilde{\omega}^{-(j)}(m) \wedge \tilde{\omega}^{-(j)}(z) = \tilde{r},$$

$$\widetilde{0.20} = \tilde{\omega}^{+(j)}(m * z) < \tilde{\omega}^{+(j)}(m) \wedge \tilde{\omega}^{+(j)}(z) = \widetilde{0.70}$$

and

$$\tilde{1} = \tilde{\pi}^{(j)}(m * z) > \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z) = \tilde{l}$$

for all $j \in \{1, 2, \dots, k\}$. That is,

$$\widetilde{[0, 0.20]} = \tilde{\omega}^{(j)}(m * z) \preceq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z) = \widetilde{[r, 0.70]}$$

and

$$\tilde{1} = \tilde{\pi}^{(j)}(m * z) > \tilde{\pi}^{(j)}(m) \vee \tilde{\pi}^{(j)}(z) = \tilde{l}.$$

Thus, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is not a C_kP subalgebra of \mathbb{D} .

Definition 6. A C_kP structure $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ of a BCI-algebra \mathbb{B} is said to be closed if conditions (1) and (2) of Definition 5 hold, and for any $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, $\tilde{\omega}^{(j)}(0 * m) \succeq \tilde{\omega}^{(j)}(m)$ and $\tilde{\pi}^{(j)}(0 * m) \leq \tilde{\pi}^{(j)}(m)$.

In the next two theorems, we discuss the relation between a C_kP subalgebra and a closed C_kP ideal in a BCI-algebra \mathbb{B} .

Theorem 14. Every closed C_kP ideal of a BCI-algebra \mathbb{B} is a C_kP subalgebra.

Proof. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a closed C_kP ideal of \mathbb{B} . Then, for all $m, z \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$,

$$\begin{aligned} \tilde{\omega}^{(j)}(m * z) &\succeq \tilde{\omega}^{(j)}((m * z) * m) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &= \tilde{\omega}^{(j)}((m * m) * z) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &= \tilde{\omega}^{(j)}(0 * z) \bar{\wedge} \tilde{\omega}^{(j)}(m) \\ &\succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(z) \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}^{(j)}(m * z) &\leq \tilde{\pi}^{(j)}((m * z) * m) \vee \tilde{\pi}^{(j)}(m) \\ &= \tilde{\pi}^{(j)}((m * m) * z) \vee \tilde{\pi}^{(j)}(m) \\ &= \tilde{\pi}^{(j)}(0 * z) \bar{\vee} \tilde{\pi}^{(j)}(m) \\ &\leq \tilde{\pi}^{(j)}(m) \bar{\vee} \tilde{\pi}^{(j)}(z) \end{aligned}$$

Hence, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of \mathbb{B} . \square

Theorems 15 and 16 state under which condition a C_kP subalgebra is a closed C_kP ideal.

Theorem 15. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of a BCI-algebra \mathbb{B} . If $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra, then $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a closed C_kP ideal of \mathbb{B} .

Proof. Assume that $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of a BCI-algebra \mathbb{B} . Then,

$$\tilde{\omega}^{(j)}(0 * m) \succeq \tilde{\omega}^{(j)}(0) \bar{\wedge} \tilde{\omega}^{(j)}(m) = \tilde{\omega}^{(j)}(m)$$

and

$$\tilde{\pi}^{(j)}(0 * m) \leq \tilde{\pi}^{(j)}(0) \bar{\vee} \tilde{\pi}^{(j)}(m) = \tilde{\pi}^{(j)}(m)$$

for all $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$. Hence, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a closed C_kP ideal of \mathbb{B} . \square

Theorem 16. In a p -semisimple BCI-algebra \mathbb{B} , every C_kP subalgebra is a closed C_kP ideal.

Proof. Assume that $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP subalgebra of a p -semisimple BCI-algebra \mathbb{B} . Then, for all $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, we obtain

$$\tilde{\omega}^{(j)}(0) = \tilde{\omega}^{(j)}(m * m) \succeq \tilde{\omega}^{(j)}(m) \bar{\wedge} \tilde{\omega}^{(j)}(m) = \tilde{\omega}^{(j)}(m)$$

and

$$\tilde{\pi}^{(j)}(0) = \tilde{\pi}^{(j)}(m * m) \leq \tilde{\pi}^{(j)}(m) \bar{\vee} \tilde{\pi}^{(j)}(m) = \tilde{\pi}^{(j)}(m).$$

Hence,

$$\tilde{\omega}^{(j)}(0 * m) \succeq \tilde{\omega}^{(j)}(0) \bar{\wedge} \tilde{\omega}^{(j)}(m) = \tilde{\omega}^{(j)}(m)$$

and

$$\tilde{\phi}^{(j)}(0 * m) \leq \tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(m) = \tilde{\pi}^{(j)}(m).$$

Now, let $m, z \in \mathbb{B}$. Then, for $j \in \{1, 2, \dots, k\}$, we obtain

$$\begin{aligned} \tilde{\omega}^{(j)}(m) &= \tilde{\omega}^{(j)}(z * (z * m)) \\ &\succeq \tilde{\omega}^{(j)}(z) \bar{\wedge} \tilde{\omega}^{(j)}(z * m) \\ &= \tilde{\omega}^{(j)}(z) \bar{\wedge} \tilde{\omega}^{(j)}(0 * (m * z)) \\ &\succeq \tilde{\omega}^{(j)}(m * z) \bar{\wedge} \tilde{\omega}^{(j)}(z) \end{aligned}$$

and

$$\begin{aligned} \tilde{\pi}^{(j)}(m) &= \tilde{\pi}^{(j)}(z * (z * m)) \\ &\leq \tilde{\pi}^{(j)}(z) \vee \tilde{\pi}^{(j)}(z * m) \\ &= \tilde{\pi}^{(j)}(z) \vee \tilde{\omega}^{(j)}(0 * (m * z)) \\ &\leq \tilde{\pi}^{(j)}(m * z) \vee \tilde{\pi}^{(j)}(z) \end{aligned}$$

Therefore, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a closed C_kP ideal of \mathbb{B} . \square

Theorem 17. In associative BCI-algebra \mathbb{B} , every C_kP subalgebra is a closed C_kP ideal.

Proof. The proof is straightforward, since every associative BCI-algebra is a p-semisimple BCI-algebra. \square

Theorem 18. Every closed C_kP ideal of a BCI-algebra \mathbb{B} is a C_kP ideal.

Proof. The proof is straightforward. \square

In the next example, we can note that the reverse direction of the above theorem does not have a role in general.

Example 4. Let $\mathbb{B} = \{2^r : r \in \mathbb{Z}\}$ be a set with the \div -operation, where \div is an ordinary division. Then, $(\mathbb{B}; \div, 1)$ is a BCI-algebra. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_3} : \mathbb{B} \rightarrow \mathbb{P}[0, 1]^3 \times [0, 1]^3$ be a C_3P structure on \mathbb{B} defined by:

$$\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_3}(r) = \left\{ \begin{array}{l} \langle ([\eta_1, \eta'_1], \varepsilon_1), ([\eta_2, \eta'_2], \varepsilon_2), ([\eta_3, \eta'_3], \varepsilon_3) \rangle, \quad r \geq 0 \\ \langle ([\zeta_1, \zeta'_1], \kappa_1), ([\zeta_2, \zeta'_2], \kappa_2), ([\zeta_3, \zeta'_3], \kappa_3) \rangle, \quad r < 0, \end{array} \right\}$$

where $[\eta_i, \eta'_i] \succ [\zeta_i, \zeta'_i]$, $\varepsilon_i < \kappa_i$, $\eta_i \leq \eta'_i$ and $\zeta_i \leq \zeta'_i$ for $i = 1, 2, 3$. Then, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_3}$ is a C_3P ideal of \mathbb{B} , but it is not closed since

$$\tilde{\omega}^{(1)}(1 \div 2^5) = \tilde{\omega}^{(1)}(2^{-5}) = [\zeta_1, \zeta'_1] \prec [\eta_1, \eta'_1] = \tilde{\omega}^{(1)}(2^5)$$

and/or

$$\tilde{\pi}^3(1 \div 2^7) = \tilde{\pi}^3(2^{-7}) = \kappa_3 > \varepsilon_3 = \tilde{\pi}^3(2^7).$$

The next Theorem states under which condition a C_kP ideal is a closed C_kP ideal.

Theorem 19. In a weakly BCK-algebra \mathbb{B} , every C_kP ideal is a closed C_kP ideal.

Proof. Let $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ be a C_kP ideal of a weakly BCK-algebra \mathbb{B} . Then, for all $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$, we obtain

$$\begin{aligned}\tilde{\omega}^{(j)}(0 * m) &\succeq \tilde{\omega}^{(j)}((0 * m) * m) \wedge \tilde{\omega}^{(j)}(m) \\ &\succeq \tilde{\omega}^{(j)}(0) \wedge \tilde{\omega}^{(j)}(m) \\ &= \tilde{\omega}^{(j)}(m)\end{aligned}$$

and

$$\begin{aligned}\tilde{\phi}^{(j)}(0 * m) &\leq \tilde{\pi}^{(j)}((0 * m) * m) \vee \tilde{\pi}^{(j)}(m) \\ &\leq \tilde{\pi}^{(j)}(0) \vee \tilde{\pi}^{(j)}(m) \\ &= \tilde{\pi}^{(j)}(m)\end{aligned}$$

Hence, $\mathcal{C}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a closed C_kP ideal of \mathbb{B} . \square

In the next two figures, we can note the correspondence between C_kP -subalgebras and (closed) C_kP ideals in (BCK)BCI-algebras, p-semisimple BCI-algebras, weakly BCK-algebras, and associative BCI-algebras.

Figure 3 shows the correspondence between C_kP subalgebras and (closed) C_kP ideals in BCK and BCI-algebras.

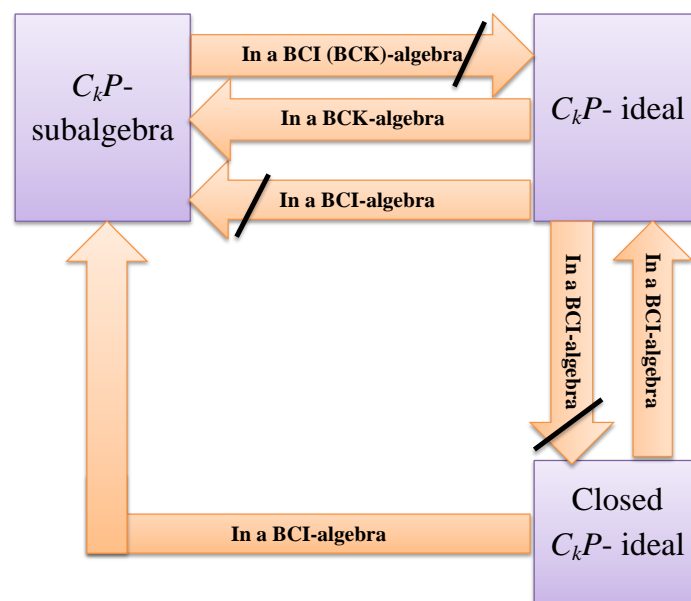


Figure 3. Algebraic relations part I.

Figure 4 shows the correspondence between C_kP subalgebras and (closed) C_kP ideals in p-semisimple BCI-algebras, weakly BCK-algebras, and associative BCI-algebras.

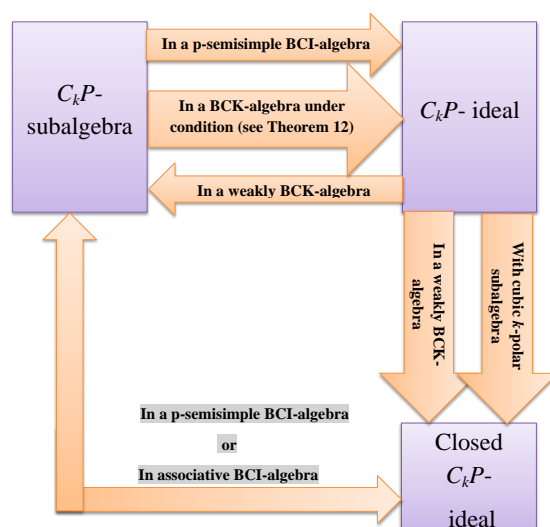


Figure 4. Algebraic relations part II.

5. Cubic k -Polar Extension Property

In the current section, we consider the C_kP extension property for a C_kP ideal.

Theorem 20 ([2]). Let K and M be two ideals of \mathbb{B} with $K \subseteq M$. If K is an ideal, then so is M .

Definition 7. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ and $\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$ be two C_kP structures of \mathbb{B} . Then, $\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$ is called a C_kP extension of $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$, symbolized by $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k} \lesssim \tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$, if $\tilde{\omega}(m) \succeq \tilde{\psi}(m)$ and $\tilde{\pi}(m) \leq \tilde{\chi}(m)$ for all $m \in \mathbb{B}$. That is, $\tilde{\omega}^{(j)}(m) \succeq \tilde{\psi}^{(j)}(m)$ and $\tilde{\pi}^{(j)}(m) \leq \tilde{\chi}^{(j)}(m)$ for all $m \in \mathbb{B}$ and $j \in \{1, 2, \dots, k\}$.

Next, we establish the C_kP extension theorem of the C_kP ideal.

Theorem 21. Let $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ and $\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$ be two C_kP ideals of \mathbb{B} such that $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k} \lesssim \tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$, and $\tilde{\omega}^{(j)}(0) \succeq \tilde{\psi}^{(j)}(0)$ and $\tilde{\pi}^{(j)}(0) \leq \tilde{\chi}^{(j)}(0)$ for each $j \in \{1, 2, \dots, k\}$. If $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} , then so is $\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$.

Proof. To prove that $\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$ is a C_kP ideal of \mathbb{B} , it is sufficient to show that, for any $[u, v] \in \mathbb{P}[0, 1]^k$ and $\tilde{\gamma} \in [0, 1]^k$, $\Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma}))$ is either empty or an ideal of \mathbb{B} . Assume that $\Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma}))$ is nonempty and $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k} \lesssim \tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$. Now, if $m \in \Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma}))$, then $\tilde{\omega}^{(j)}(m) \succeq [u_j, v_j]$ and $\tilde{\pi}^{(j)}(m) \leq \gamma_j$ for all $j \in \{1, 2, \dots, k\}$. Thus, $\tilde{\psi}^{(j)}(m) \succeq [u_j, v_j]$ and $\tilde{\chi}^{(j)}(m) \leq \gamma_j$ for all $j \in \{1, 2, \dots, k\}$, that is, $m \in \Theta(\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}; ([u, v], \tilde{\gamma}))$. Thus,

$$\Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma})) \subseteq \Theta(\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}; ([u, v], \tilde{\gamma})).$$

By the hypothesis, $\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}$ is a C_kP ideal of \mathbb{B} , and it follows from Theorem 8 that $\Theta(\tilde{\mathcal{C}}_{(\tilde{\omega}, \tilde{\pi})_k}; ([u, v], \tilde{\gamma}))$ is an ideal of \mathbb{B} . By Theorem 20, $m \in \Theta(\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}; ([u, v], \tilde{\gamma}))$ is also an ideal of \mathbb{B} . Thus, by using Theorem 9, we obtain that $\tilde{\mathcal{C}}_{(\tilde{\psi}, \tilde{\chi})_k}$ is a C_kP ideal of \mathbb{B} . \square

6. Conclusions

The CS, which was presented by Jun et al., is a generalization of the *IFS*, that is, Jun's and co-workers CSs are an extension of Atanassov's *IFSs*. In this paper, we originated the C_kP subalgebras and (closed) C_kP ideals in BCK(BCI)-algebras and studied some dominant properties of these concepts in detail. We gave a fundamental bridge between crisp subalgebras/ideals and C_kP subalgebras/ideals. In addition, we discussed relations between C_kP subalgebra, C_kP ideals and closed C_kP ideals. We provided a condition for a C_kP subalgebra to be a C_kP ideal in a BCK-algebra. In a BCI-algebra, we provided conditions for a C_kP subalgebra to be a C_kP ideal, and conditions for a C_kP subalgebra to be a closed C_kP ideal. We proved that, in a weakly BCK-algebra, every C_kP ideal is a closed C_kP ideal. Finally, we established the C_kP extension property for a C_kP ideal and gave some illustrative examples to support the results in this paper.

The findings of this work can be applied to a variety of algebraic structures—for instance, semigroups, Γ -semigroups, fields, normed rings, and hemirings (see, [36–41]). Furthermore, the conception of a C_kP structure used in this article can be analyzed according to the ideas in [42–45], which will pave the way for a lot of future research.

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